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Fuzzy Bi-Topological Space And Separation Axioms

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Abstract : We deal with Fuzzy bi-topological Space and Separation Axioms. In this paper, we introduced the concept of Fuzzy Bi-topological Space which is a non empty Set X equipped with two fuzzy topologies on it and different pair wise separation Axioms are defend as generalization of natural Separation axioms.

Introduction :

A fuzzy bi-topological space is a non-empty Set X equipped with two fuzzy topologies on it. Different pairwise separation axioms are defined as generalization of natural separation axioms in the sense that such a notion reduces to the natural separation axioms of a fuzzy topological space when two topological spaces coincide. In this paper, pairwise, separation axioms are introduced and a mixed topology is introduced with the help of two fuzzy topologies of a fuzzy bi-topological space. Relation between such pairwise separation axioms and natural fuzzy separation axioms of the mixed fuzzy topological space are investigated. Finally, pairwise fuzzy normal bi-topological space, pairwise weakly and pairwise strongly separated space are introduced and investigated their properties with the mixed topology.

3. Preliminaries :- For an easy understanding of the material incorporated in this paper, we reproduce the following definitions and results which can be found in any standard textbook on fuzzy topological space.

4. Fuzzy Bi-topological Space : To cope up with the material incorporated in this paper, we need some rudiments of fuzzy topological space. We follow the terminology and the results of the paper [12], [53], [93] and [97]. Now we define separation axioms in a fuzzy bi-topological space.

Definition 4.1 : A fuzzy bi-topological space space (X, $\Im_1$, $\Im_2$) is said to be pairwise T_1 if of every pair of distinct fuzzy points x and y in X, there exits $\Im_1$ open set U and a $\Im_2$ open set V. s.t. U (x) = 1, y $\notin$ V and x $\notin$ V, V (y) = 1.

Definition 4.2 : A fuzzy bi-topological space (X, $\Im_1$, $\Im_2$) is said to be pairwise fuzzy Hausdorff space if for each pair of distinct points x and y, there are $\Im_1$ open set U and a $\Im_2$ open set V s.t. U (x) = 1, V (y) = 1 and U $\cap$ V = 0.

Definition 4.3 : A fuzzy bi-topological space (X, $\Im_1$, $\Im_2$) is called – pairwise fuzzy regular w.r.t. $\Im_2$ iff $\alpha \in (0, 1)$, U $\in$ $\Im_1^c$, x $\in$ X and $\alpha <$ 1-U (x) imply that there exits V$\in\Im_2$ and W $\in\Im_2$ with $\alpha <$ V (x) V $\subseteq$ V and V $\subseteq$ 1-W. (X, $\Im_1$, $\Im_2$) is called pairwise fuzy regular if it is – fuzzy regular with respect to $\Im_1$ and $\Im_2$ fuzzy regular with respect to $\Im_1$.

The following theorem plays a key role in the sequel. It is a relation between compactness and closeness of a subject of a pairwise Hausdorff bi-topological space. The ordinary subset Y is regarded as a fuzzy subset.

Definition 4.4: If (X, $\Im_1$, $\Im_2$) is a pairwise Hausdorff fuzzy bi-topological space and Y is an ordinary -- 1 compact fuzzy set in X then Y is -- closed.

Proof : It is sufficient to show x_λ is not in Y implies x_λ is not an accumulation point of Y. $x_\lambda \notin$ Y means 1>λ>y (x) and therefore, x $\notin$ Y. So, x $\neq$ Y for all y $\in$ Y. by the pairwise Hausdorff character of (X, $\Im_1$, $\Im_2$) there exists a $\Im_1$ open set U and a $\Im_2$ open set V s.t. U (x) = V (y) and U $\cap$ V = 0. Thus, for $x_\lambda \notin$ Y and y $\in$ Y. U (x) = (V$\cap$Y) = 1 with U $\cap$ V = 0. Varying y over all points belonging to Y, the collection {Y$\cap$V} is an 1^*-shading of Y. so, it is reducible to a finite 1^* -subshading say, {Y$\cap V_{y1,}$ Y$\cap V_{y2}$, ……… Y$\cap V_{yn}$ }. We write V = V_{y1}

Since, $U^{y1}{}_x$ (x) = 1 for all 1$\leq$ i $\leq$ n, we have U (x) = 1 and $x_\lambda \in$ U. Now U $\cap$V = ($U^{y1}{}_x$ $\cap$ ……, $\cap$ $U^{yn}{}_x$) $\cap$ ($V_{y1}\cup$ $V_{y2}\cup$ ….. $\cup$ V_{yn}) =0 for y $\in$ Y, there exists Y$\cap V_{y1}$ s. t. Y$\cap$ V_{y1} (y) =1 implies Y $\cap$ V (y) =1=Y (y) therefore Y $\cap$ V =Y. Also, Y$\cap$U= Y$\cap$V$\cap$U=0 implies that Y (x) =0 or U (x) =0. Therefore, Y (x) + U (x) >1. Hence Y and U are not quasi-coincident, and therefore x_λ is not a $\Im_1$ – accumulation point of Y. This proves that Y is $\Im_1$ closed.

4.5 Definition : In a pairwise fuzzy Hausdorff space (X, $\Im_1$, $\Im_2$) is $\Im_1$ – 1* compact subset is $\Im_2$ –closed.

4.6 Definition : With the help of two fuzzy topologies of a fuzzy bi-topological space a thirdb fuzzy topology is defined on it. This topology is named as mixed fuzzy topology. We then relate separation axioms relative to the mixed topology with pairwise separation axioms of the fuzzy bi-topological space.

4.7 Definition : Let (X $\Im_1$, $\Im_2$) be a fuzzy bi-topological space, {Y_α} be a collection of ordinary subsets of X which are $\Im_2$ 1* - compact as fuzzy subsets.

Let, τ ={i_α,: Y → X} and ($\Im_\beta$) be the collection of fuzzy topologies on X s. t.

$I_\alpha : (Y_\alpha, \Im_1) \rightarrow (X, \Im_\beta)$ are continuous where $(Y_\alpha, \Im_1)$ means subspace topology on X s.t. ia are continuous. That is, $\Im_1$ $(\Im_2)$ is topology s.t.

(a) $\Im_1$ $(\Im_2) \supseteq \Im_\beta$, for all β s. t. $i_\alpha : (Y_\alpha, \Im_1) \rightarrow (X, \Im_1)$ $(\Im_2))$ are continuous.

(b) If $(\Im_0) \supseteq \Im_\beta$, for all β and $i_\alpha : (Y_\alpha, \Im_1) \rightarrow (X, \Im_0)$ are continuous then $\Im_0 \supseteq \Im_1$ $(\Im_2)$

The fuzzy topology $\Im_1$ $(\Im_2)$ is called a mixed fuzzy topology on X. Clearly, $\Im_1 \in \{\Im_\beta\}$ and therefore, $\Im_1 \subseteq \Im_1$ $(\Im_2)$. Although we have used the symbol $\Im_1$ $(\Im_2)$ for the mixed topology arising out of fuzzy topologies $\Im_1$ and $\Im_2$. This mixed topology $\Im_1$ $(\Im_2)$, is not the same as $\Im_1(\Im_2)$, of the preceding paper Cf 3.2.1). The theorem 4.2 is applied to the rest of this paper. The following theorem shows that relation between Hausdorff character of the mixed topology and the pairwise Hausdorff character of the bi-topological space.

<u>4.8 Definition</u> : If $(X\ \Im_1\ , \Im_2)$ is a pairwise fuzzy Hausdorff space then the mixed fuzzy topology $\Im_1$ $(\Im_2)$ is a fuzzy Hausdorff topology.

<u>Proof</u> :- Since $(X\ \Im_1\ , \Im_2)$ is a pairwise fuzzy Hausdorff space, x, y $\in$ X and x $\neq$y there exist $U\in\Im_1$ and $V\in\Im_2$ U (x) = V (y) s.t. $U\cap V = 0$. To prove that $(X, \Im_1\ , (\Im_2))$ is a fuzzy Hausdorff space, we claim that both U and V are $\Im_1$, $(\Im_2)$ open. Let, Y_α be ordinary subsets of X which are 1*-compact w.r.t. fuzzy topology $\Im_2$ Let, □ $=\{i_\alpha: Y \rightarrow X\}$ and $(\Im_\beta)$ be the collection of inclusion mappings and fuzzy topologies on X s.t. $i_\alpha : (X, \Im_1\ , (\Im_2)) \rightarrow (X, \Im_\beta)$ are continuous. For each $z\in Y_\alpha$, i_α^{-1} (U) (z) = U $((i_\alpha (z) = \min \{Y_\alpha (z), U(z)\} = (Y_\alpha \cap U)$ (z)

Therefore, i_α^{-1} (U) is $(Y_\alpha, \Im_1)$ open Since $\Im_1$ is coarser than $\Im_1$ $(\Im_2)$ $\Im_1$-open set U is $\Im_1$ $(\Im_2)$-open. Similarly, K= i_α^{-1} (v) = $Y_\alpha \cap V$ is open in $(Y_\alpha, \Im_2)$ and therefore its complement in Y_α, Y_α -K is closes $(Y_\alpha, \Im_2)$. Application of proposition 1.7.9 shows that Y_α –K is $(Y_\alpha, \Im_2)$ – 1*-compact. Also $(Y_\alpha, \Im_1\ \Im_2)$ inherits pairwise Hausdorff character from $(X, \Im_1\ , \Im_2)$. Then by theorem 4.2.4, Y_α –K is $(Y_\alpha, \Im_1)$ closed and i_α^{-1} (v) = $Y_\alpha \cap v$= K is $(Y_\alpha, \Im_1)$ open for every Y_α. We claim that $v\in\Im_1$, $(\Im_2)$. Let $\Im_0 = \{v \mid i_\alpha^{-1} (v) \in (Y_\alpha, \Im_1)$ open for every Y_α we claim that $v\in\Im_1$ $(\Im_2)$. Let $\Im_0 = \{v \mid i_\alpha^{-1} (v) \in (Y_\alpha, \Im_1)$ for all $Y_\alpha\}$Now $\Im_0$ is a topology on X s.t. i_α $((X, y_\alpha , \Im_1)) \rightarrow ((X, \Im_0)$ are continuous So, $\Im_0$ is one of the members of $\{\Im_\beta\}$ and hence $\Im_0$ is $\subseteq\Im_1$ $(\Im_2)$. Now , K= $Y_\alpha \cap V = i_\alpha^{-1}(v) \in (Y_\alpha, \Im_1)$ for all Y_α So, $V\in\Im_0 \subseteq \Im_1$ $(\Im_2)$ Now, K= $Y_\alpha \cap V = i_\alpha^{-1}(v) \in (Y_\alpha, \Im_1)$ for all Y_α So, $V\in\Im_0 \subseteq \Im_1$ $(\Im_2)$. This prove that $V\in\Im_1$ $(\Im_2)$. Thus x,y$\in$X and x$\neq$ y implies that there exists $U\in\Im_1$ $(\Im_2)$ and $V\in\Im_1$ $(\Im_2)$ with U(x) =V(y)=1 and $u\cap v=0$. Therefore , $(X, \Im_1\ , (\Im_2))$ is a fuzzy Hausdorff space.

<u>4.9 Definition</u> : Several authors have studied fuzzy regularity in different ways. Some of which are equivalent and others are independent as shown by Dewan M. Ali [35]. The following lemmas are associated with the theorem of pairwise fuzzy regular bi-topological spaces.

4.10 Definition : If U is a closed relative to subspace topology on y induced from then $\Im$, $U=Y\cap U_0$ where V_0 is $\Im$-closed.

Proof :-Here the subspace topology $\Im_{1y}=\{Y\cap G\,|\,G\in\Im\}$. U is $\Im_{1y}$ –closed. So 1-U is $\Im_{1y}$ –open. $1-U=Y\cap V$ where V is $\Im$-open. Therefore 1-U(y)=min {Y(y), V(y)}= min {1, V(y)}=V(y) implies that U(y)=1-V(y)=min {Y(y), 1-V(y)} = $(Y\cap V^c)$ (y). Therefore $U=(Y\cap V^c)=Y\cap U_0$. Since V is $\Im$-open, $V^c=U_0$ is $\Im$-closed. Here is the representation of open subsets of the supremum topology $V\Im_\alpha$ defined in 4.3.1.

4.11 Definition : V $\Im_\alpha$–open sets are Union of finite intersections of different $\Im_\alpha$–open sets

Proof :- Let $\Im$ be the collection of Unions of finite intersections of members $U\Im_\alpha$, that is $\Im=\{U\,(\bigcap_{i=1}^{n} U_i)$, $U_i\in U\ \Im_\alpha$

Clearly $\Im$ is a fuzzy topology and (i) $\Im\supseteq\Im_\alpha$ for all α

(ii) If $\Im_0$ is fuzzy in $\Im_0\supseteq\Im_\alpha$ for all α, then, $\Im_0\supseteq\Im\supseteq\Im_\alpha$. Let $U(\bigcap_{i=1}^{n} U_i)\in\Im$, Where $U_i\in U\ \Im_\alpha$ then $U_\alpha\ (\bigcap_{i=1}^{n} U_i)\notin\Im$,

$\Im_0$ for some $i=i_0\Rightarrow U\notin\Im_\alpha$ for all α, which contradicts that $U\in U\Im_\alpha$. Hence $U_{n_i}(\bigcap_{i=1}^{n} U_i)\in\Im_0$ Therefore, $\Im\subseteq\Im_0$ and $\Im$=1.u.b. $\Im_0$.

4.12 Definition : f: $(X, \Im) \rightarrow (Y, \Im_\alpha)$ is continuous for all if f: $(X, \Im) \rightarrow (Y, V\Im_\alpha)$ is continuous.

Proof :- (i) Let f: $(X, \Im) \rightarrow (Y, \Im_\alpha)$ is continuous for all α

$U\in V\Im_\alpha$, then $U= U\,(\bigcap_{i=1}^{n} U_i)$ Now $f^{-1}\,(U\,(\bigcap_{i=1}^{n} U_i) = U\,(\bigcap_{i=1}^{n} f^{-1}U_i)\in\Im$. Hence f is continuous from $(X, \Im)$ to $(Y, \Im_\alpha)$

for all α.

(ii) Let f: $(X, \Im) \rightarrow (Y, V\Im_\alpha)$ be continuous Every U in $\Im_\alpha$ is in $V\Im_\alpha$ and hence

$f^{-1}(U)\in\Im$. Hence f: $(X, \Im) \rightarrow (Y, \Im_\alpha)$ is continuous.

4.13 Definition : Let ((X, $\Im$, $\Im$) be pairwise fuzzy Hausdorff and pairwise fuzzy regular space, Y_k be $\Im_2$-1*-compact ordinary subsets of X when regarded as fuzzy subsets and $\Im_1(\Im_2)_{1yk}$ is fuzzy regular for each Y_k

Proof :- Suppose $x\in Y_k$ and $U\in\Im_1(\Im_2)_{1yk})^C$ and $\alpha\in(0,1)$, s.t. $\alpha<1-U$

(x). Then by lemma 4.4.1, $U=Y_k \cap U_1$ where U_1 is $\mathfrak{I}_1(\mathfrak{I}_2)$ – closed. By the continuity of $i_{yk} : (Yk, \mathfrak{I}_1) \rightarrow (X\, \mathfrak{I}_1 , \mathfrak{I}_2))$, i^{-1}_{yk} (U_1) $\mathfrak{I}_{1\ 1yk}$ closed, i.e. $U=Y_k \cap U_1$ is $\mathfrak{I}_{1\ 1yk}$ –closed. Since Y_k is $\mathfrak{I}_1$-1*-compact set in the pairwise Hausdorff space $(X, \mathfrak{I}_1, \mathfrak{I}_2)$, by theorem 4.2.4, Y_k is $\mathfrak{I}_1$-closed. Therefore by lemma 4.4.1 U is $\mathfrak{I}_1$-closed. Since $(X, \mathfrak{I}_1, \mathfrak{I}_2)$ is pairwise fuzzy regular there exists $V \in \mathfrak{I}_1$ and $W \in \mathfrak{I}_2$ with $\alpha < V(x)$ $U \subseteq W$ and $V \subseteq 1-W$ Also $V \in \mathfrak{I}_1 \subseteq \mathfrak{I}_1$ $(\mathfrak{I}_2)$ implies $V \in \mathfrak{I}_1 \subseteq \mathfrak{I}_1$ $(\mathfrak{I}_2)$. It can be shown that $W \in \mathfrak{I}_1$ $(\mathfrak{I}_2)$ as argued in proposition 4.3.2. We have $U \subseteq Y_k \cap W$, $\alpha < Y_k \cap V)$ (x) and $Y_k \cap V \subseteq 1-Y_k \cap W$. Therefore, $(Y_k, \mathfrak{I}_1 , \mathfrak{I}_2))$, i^{-1}_{yk} is fuzzy regular.

4.14 Definition : Let $(X, \mathfrak{I}_1 , \mathfrak{I}_2)$ be a pairwise fuzzy Hausdorff space in which every $\mathfrak{I}_2$-1*-compact sets are $\mathfrak{I}_2$-1*-compact. Let Y_k $\mathfrak{I}_2$-1*-compact ordinary sets and $\mathfrak{I}_1(\mathfrak{I}_2)$ be the mixed topology on X. If $(X, \mathfrak{I}_1 , \mathfrak{I}_2))$ is a fuzzy regular space then $Y_k , \mathfrak{I}_1 i^{-1}_{yk} , \mathfrak{I}_2 i^{-1}_{yk}$) is pairwise fuzzy regular for each Y_k.

Proof :- Suppose $x \in Y_k$ U is a $\mathfrak{I}_1$ i^{-1}_{yk} closed set and $U=Y_k \cap U_0$ where U_0 is $\mathfrak{I}_1$–closed and Y_k is $\mathfrak{I}_1$-closed [Cf 4.2.4]. Then U is $\mathfrak{I}_1$ $(\mathfrak{I}_2)$-closed. Since $(X, \mathfrak{I}_1 , \mathfrak{I}_2))$ is fuzzy regular, for $\alpha \in (0,1)$, $U \in (\mathfrak{I}_1 , \mathfrak{I}_2))^C$, $x \in X$ and $\alpha < 1-U$ (x) there exist V, $W \in \mathfrak{I}_1 , \mathfrak{I}_2)$ with $\alpha < 1-V(x)$, $U \subseteq W$ and $V \subseteq 1-W$. Now i^{-1}_{yk} (v) = $Y_k \cap V$ is $\mathfrak{I}_{1\ 1yk}$ – open and hence $[1-Y_k \cap V]$ is $\mathfrak{I}_{1\ 1yk}$ closed. Since Y_k is $\mathfrak{I}_1$-1*-compact, $[1-(Y_k \cap V)]$ is $\mathfrak{I}_{2\ 1yk}$ closed and hence $Y_k \cap V$ is $\mathfrak{I}_{2\ 1yk}$ open and $U \subseteq Y_k \cap W$. Now i^{-1}_{yk} (w) $=Y_k \cap W$ is $\mathfrak{I}_{2\ 1yk}$ open. Therefore $Y_k \cap V \in \mathfrak{I}_{2\ 1yk}$ and $Y_k \cap W \in \mathfrak{I}_{2\ 1yk}$, $Y_k \cap V \subseteq 1-$ $(Y_k \cap W)$. So, $(Y_k, \mathfrak{I}_{1\ 1yk}, \mathfrak{I}_{2\ 1yk})$ $\mathfrak{I}_{2\ 1yk}$ is regular w.r.t. . Hence $(Y_k, \mathfrak{I}_{1\ 1yk}, \mathfrak{I}_{2\ 1yk})$ is pairwise fuzzy regular for each Y_k. This completes the result.

5. Conclusion :- The results in this paper gives the structural properties of a Fuzzy bi-topological space and pairwise separation axioms as generalization of natural separation axioms. Many more informations regarding its structural properties and applications can be expected.

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REFERENCES

1. Arya S P, Nour T M, Separation Axioms for bi-topological spaces, *Jour. Pure Appl. Math* 19(1) 42-50, 1998
2. Chang, C L, Fuzzy topological spaces, *J Math Anal Appl* 24, 182-192, 1968
3. Das N R, Baishya, P C, Fuzzy bi-to pological space and separation Axioms *The Jour, Fuzzy Math.* 2 3890-386, 1994
4. Ganguly S, Saha, S on separation axioms and separation of connected sets in Fuzzy topological space *Bull cal. Math Soc.* 79 215-225 1987
5. Kelley J C, Bi-topological Spaces *Inoc London Math.* Soc 3 (13) 781-89 1963
6. Sinha S P, Separation axioms in fuzzy topological spaces, *Fuzzy sets and systems* 45, 61-270, 1992

7. Srivastava A K Dewan, M Ali A note on K K Azad's Fuzzy Hausdorffness conceptsn, *Fuzzy sets and systems* 42 363-367, 1991
8. Das N R, Baishya, P C Mixed topo logical spaces, *The Journal of fuzzy math* Vol. 3 No 4 December 1995
9. Ganguly, S Singha D, Mixed topology for a bi-topological space, Bull. Cal. Math Soc. 76 (1984), 304-314
10. Srivastava, R Lal, S N Srivastava A. K. Fuzzy T1 topological spaces, *J Math Anal. Appl.* 102, 442-448, 1984
11. Dewan, M. Ali : A note on fuzzy regu larity Concepts, fuzzy sets and systems 35 101-104, 1990

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